Name of College –

S.S. College J-Bad

Subject – Mathematics

Topic → Problem based on Tangent and Normal

Date – 13-07-2020

Time – 10 to 11.30

Problem – Tangent at any point of the curve $\left(\frac{x}{a}\right)^{2/3} + (y/b)^{2/3} = 1$ meets the co-ordinate axes in A and B. Show that the locus of the point with (OA, OB) as a co-ordinate is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution Equation of the Given Curve is

$$\left(\frac{x}{a}\right)^{2/3} + (y/b)^{2/3} = 1 \longrightarrow (1)$$

∴ Parametric equation of (1)

$$x_1 = a\cos^3\theta \quad y_1 = b\sin^3\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{\cancel{3}b\sin^2\theta\cos\theta}{\cancel{3}a\cos^2\theta\sin\theta} = -\frac{b\sin\theta}{a\cos\theta}$$

∴ Equation of Tangent at (x_1, y_1) is

$$\frac{y-y_1}{x-x_1} = (dy/dx)_{x_1,y_1}$$

$$\Rightarrow \frac{y-b\sin^3\theta}{x-a\cos^3\theta} = -\frac{b\sin\theta}{a\cos\theta}$$

$$\Rightarrow a\cos\theta\, y - ab\sin^3\theta\cos\theta = -b\sin\theta\, x + ab\sin\theta\cos^3\theta$$

$\Rightarrow b\sin\theta\, x + a\cos\theta\, y = ab\sin\theta\cos\theta$

$\Rightarrow \dfrac{b\sin\theta\, x}{ab\sin\theta\cos\theta} + \dfrac{a\cos\theta}{ab\sin\theta\cos\theta} y = 1$

$\Rightarrow \dfrac{x}{a\cos\theta} + \dfrac{y}{b\sin\theta} = 1$

$\Rightarrow OA = a\cos\theta$

$OB = b\sin\theta$

Now we have to find the locus of (OA, OB)

Let $OA = x = a\cos\theta$ $OB = y = b\sin\theta$

$\Rightarrow \dfrac{x}{a} = \cos\theta$ $y/b = \sin\theta$

$\therefore \cos^2\theta + \sin^2\theta = 1$

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$\therefore$ Locus of (OA, OB) is

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

2. Show that the portion of tangent to the curve

$x^{2/3} + y^{2/3} = a^{2/3}$

which is intercepted between the axes, is of constant length and find the area of the portion included between the axes and the tangent $\rightarrow$

Solution Since

Equation of tangent at (x_1, y_1) is $(x - x_1) f_x + (y - y_1) f_y = 0$

Here. Equation of Curve

$$f(x,y) = 0$$

$$\Rightarrow f(x,y) = x^{2/3} + y^{2/3} - a^{2/3} = 0$$

$$f_x = \frac{2}{3} x^{-1/3}$$

$$f_y = 2/3\, y^{-1/3}$$

$$(x - x_1)\, 2/3\, x_1^{-1/3} + (y - y_1)\, 2/3 \cdot y_1^{-1/3} = 0$$

$$\Rightarrow \frac{2}{3} x\, x_1^{-1/3} + \frac{2}{3} y\, y_1^{-1/3} = \frac{2}{3} x_1 \cdot x_1^{-1/3} + \frac{2}{3} y_1 \cdot y_1^{-1/3}$$

$$= \frac{2}{3}\left[x_1^{1-1/3} + y_1^{1-1/3}\right]$$

$$\frac{2}{3}\left[\frac{x}{x_1^{1/3}} + \frac{y}{y_1^{1/3}}\right] = 2/3\left[x_1^{2/3} + y_1^{2/3}\right]$$

$$= \frac{2}{3} \cdot a^{2/3}$$

$$\Rightarrow \frac{x}{x_1^{1/3}} + \frac{y}{y_1^{1/3}} = a^{2/3}$$

$$\Rightarrow \frac{x}{a^{2/3} \cdot x_1^{1/3}} + \frac{y}{a^{2/3} \cdot y_1^{1/3}} = 1$$

$$\therefore OA = a^{2/3} \cdot x_1^{1/3}$$

$$OB = a^{2/3}\, y_1^{1/3}$$

$$AB = \sqrt{OA^2 + OB^2}$$

$$= \sqrt{a^{4/3} x_1^{2/3} + a^{4/3} y_1^{2/3}}$$

$$= a^{2/3}\sqrt{x_1^{2/3} + y_1^{2/3}}$$

$$= \cancel{a^{2/3}\sqrt{a^{2/3}}}$$

$$= a^2 .$$

$$= \sqrt{a^{4/3} \cdot a^{2/3}}$$

$$= \sqrt{a^{6/3}}$$

$$= \sqrt{a^2}$$

$$= a$$

Which is constant.

Now Area of $\triangle OAB$.

$$= \frac{1}{2} OA \cdot OB$$

$$= \frac{1}{2} \cdot a^{2/3} \cdot x_1^{1/3} \cdot a^{2/3} \cdot y_1^{1/3}$$

$$= \frac{1}{2} a^{4/3} x_1^{1/3} \cdot y_1^{1/3}$$

Find the Equation to the tangent to

$x^{2/3} + y^{2/3} = a^{2/3}$ at the point $(a\cos^3\theta, a\sin^3\theta)$.

If P and Q be points corresponding to θ, $\pi/2 + \theta$ and the tangents at these points meet at T. Prove that

$$OP^2 + OQ^2 + 3OT^2 = \text{const.}$$

Solution → By the question Equation of Curve is

$$x^{2/3} + y^{2/3} = a^{2/3}$$

Now Tangent at (x_1, y_1) is

$$\frac{y - y_1}{x - x_1} = \left(\frac{dy}{dx}\right)_{x_1, y_1} \quad \text{———} (1)$$

$\therefore x_1 = a\cos^3\theta$ $\quad \left(\frac{dx_1}{d\theta}\right)_{(x_1, y_1)}$ $\quad$ B

$y_1 = a\sin^3\theta$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \cancel{-3a\cos^2\theta\sin\theta}$$

$$= -\frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta}$$

$$= -\frac{\sin\theta}{\cos\theta}$$

$\therefore$ equation of tangent.

$$\frac{y - a\sin^3\theta}{x - a\cos^3\theta} = -\frac{\sin\theta}{\cos\theta}$$

$$y\cos\theta - a\sin^3\theta\cos\theta = -x\sin\theta + a\cos^3\theta\sin\theta$$

$$\Rightarrow x\sin\theta + y\cos\theta = a\sin\theta\cos\theta \quad \text{——— (I)}$$

$\therefore$ This is equation of tangent at P

Now the equation of tangent at Q.

$$x\sin(\pi/2+\theta) + y\cos(\pi/2+\theta) = a\sin(\pi/2+\theta)\cos(\tfrac{\pi}{2}+\theta)$$

$$x\cos\theta - y\sin\theta = -a\cos\theta\sin\theta$$

(I) $\times \sin\theta$ + (II) $\times \cos\theta$

$$x = a\sin^2\theta\cos\theta - a\cos^2\theta\sin\theta$$

$$= a\sin\theta\cos\theta(\sin\theta - \cos\theta)$$

(I) $\times \cos\theta$ − (II) $\times \sin\theta$

$$y = a\sin\theta\cos\theta(\sin\theta + \cos\theta)$$

Now $OP^2 + OQ^2 + 3OT^2$

$$= (a\cos^3\theta - 0)^2 + (a\sin^3\theta - 0)^2 + (-a\sin^3\theta)^2 + (-a\cos^3\theta)^2 + 3\left[\{a\sin\theta\cos\theta(\sin\theta - \cos\theta)\}^2 + \{a\sin\theta\cos\theta(\sin\theta + \cos\theta)\}^2\right]$$

$$= a^2\cos^6\theta + a^2\sin^6\theta + a^2\sin^6\theta + a^2\cos^6\theta + 3a^2\sin^2\theta\cos^2\theta(\sin\theta - \cos\theta)^2 + 3a^2\sin^2\theta\cos^2\theta(\sin\theta + \cos\theta)^2$$

$$= 2a^2\left[\cos^6\theta + \sin^6\theta + 3\sin^2\theta\cos^2\theta\right]$$

$$= 2a^2(\cos^2\theta + \sin^2\theta)^3$$

$= 2a^2$ which is independent of θ and hence is a constant.